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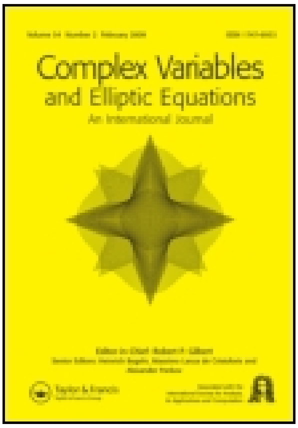

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Necessary and sufficient conditions for univalent functions

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Necessary and sufficient conditions for univalent functions

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Let $\mathcal{A}$ be the class of analytic functions in the unit disc with the normalization $f(0)=f'(0)-1=0$. This article analyses various necessary and sufficient coefficient conditions for functions $f \in \mathcal{A}$ of the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

to be univalent. We present an interesting class of univalent functions associated with the zeta function and also pose an open problem.

Keywords: coefficient inequality; analytic; Hadamard convolution; univalent and starlike functions; zeta function

AMS Subject Classifications: Primary: 30C45; Secondary: 30C20, 30C75, 30C80

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1. Introduction and main results

Let $\mathcal{A}$ denote the collection of all analytic functions f on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ normalized by the conditions $f(0)=0=f'(0)-1$, and let

$$\mathcal{S} = \{f \in \mathcal{A} : \ f \text{ is one-to-one in } \mathbb{D}\}.$$

For $f \in \mathcal{A}$ and $f(z) \neq 0$ for $0 < |z| < 1$, consider

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{1}$$

Obviously, such a representation is valid for functions $f \in \mathcal{S}$. A wellknown area theorem [1, Theorem 11 on p. 193 of Vol. 2] shows that if $f \in \mathcal{S}$ has the form (1), then

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1. \tag{2}$$

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It is natural to ask whether (2) is sufficient for univalence of the corresponding f. In Theorem 1.3, we show that the condition (2) is actually not sufficient for univalence and so, the radius of univalence is obtained for f satisfying the condition (2). For our investigation, we introduce the class $\mathcal{U}$ of all functions $f \in \mathcal{A}$ satisfying the condition

$$|\mathcal{U}_f(z)| \leq 1, \quad \mathcal{U}_f(z) = f'(z)\left(\frac{z}{f(z)}\right)^2 - 1, \quad \text{for } z \in \mathbb{D}.$$

Functions in $\mathcal{U}$ are known to be univalent in $\mathbb{D}$, see [2,3]. We refer to [4] for other related studies concerning the class $\mathcal{U}$. Now, we present a sufficient condition for univalence in terms of the coefficients b_n of the function f.

THEOREM 1.1 *Let $f \in \mathcal{A}$ and have the form* (1). *If f satisfies the condition*

$$\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1, \tag{3}$$

then $f \in \mathcal{U}$. The constant 1 *is the best possible in the sense that it cannot be replaced by a larger number.*

Proof The condition (3) implying $f \in \mathcal{U}$ is well-known [5,6] and so, it remains to prove the sharpness. Indeed, from the representation of f and the coefficient condition (3), it follows that

$$|\mathcal{U}_f(z)| = \left|-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1\right| = \left|-\sum_{n=1}^{\infty}(n-1)b_n z^n\right| \leq \sum_{n=1}^{\infty}(n-1)|b_n| \leq 1,$$

which implies that $f \in \mathcal{U}$. The proof of the first part follows.

In order to prove the second part, it suffices to show that there exist an $\varepsilon > 0$ and f of the form (1) such that

$$\sum_{n=2}^{\infty}(n-1)|b_n| = 1 + \varepsilon,$$

but $f \notin \mathcal{S}$. Now, let $f(z) = z - az^2$, where $a = \frac{\sqrt{1+\varepsilon}}{1+\sqrt{1+\varepsilon}}$ with $\varepsilon > 0$. Then $a \in (1/2, 1)$ and

$$\frac{z}{f(z)} = \frac{1}{1-az} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

where $b_n = a^n$. Thus,

$$\sum_{n=2}^{\infty}(n-1)b_n = \sum_{n=2}^{\infty}(n-1)a^n = \frac{a^2}{(1-a)^2} = 1 + \varepsilon.$$

On the other hand, $f'(z) = 1 - 2az$ and therefore, $f'(x_0) = 0$ at $x_0 = \frac{1}{2a} \in (1/2, 1)$ showing that f is not univalent in the unit disc $\mathbb{D}$. Thus, the constant 1 in the coefficient inequality (3) is the best possible. ∎

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The coefficient condition (3) is only sufficient for f to be in the class $\mathcal{U}$, but is not a necessary condition. For instance, consider the function f given by

$$\frac{z}{f(z)} = 1 + \frac{1}{3}z^2 + \frac{\sqrt{5}}{6}iz^3 + \frac{1}{9}z^4.$$

We note that

$$\left|\frac{z}{f(z)}\right| \geq 1 - \frac{1}{3}|z|^2\left|1 + \frac{\sqrt{5}}{2}iz + \frac{1}{3}z^2\right| \geq 1 - \frac{1}{3}\left(1 + \frac{\sqrt{5}}{2} + \frac{1}{3}\right) > 0$$

and so, $z/f(z)$ is non-vanishing in the unit disc $\mathbb{D}$. Also,

$$|\mathcal{U}_f(z)| = \frac{1}{3}\left|-z^2(1 + \sqrt{5}iz + z^2)\right| = \frac{1}{3}|z|^2|z + i(\sqrt{5}+3)/2|\,|z + i(\sqrt{5}-3)/2|.$$

Next, if $\psi(z) = 1 + \sqrt{5}iz + z^2$ then ψ is univalent in $\mathbb{D}$ with $\psi(0) = 1$, and

$$\max_{|z|=1}|\psi(z)| = \max_{0\leq\theta<2\pi}|2\cos\theta + \sqrt{5}i| = \max_{0\leq\theta<2\pi}\sqrt{4\cos^2\theta + 5} = 3.$$

This observation shows that $|\mathcal{U}_f(z)| < 1$ for $z \in \mathbb{D}$. On the other hand,

$$\sum_{n=2}^{\infty}(n-1)|b_n| = \frac{1}{3} + \frac{\sqrt{5}}{3} + \frac{1}{3} > 1$$

and the claim is proved.

Similar to the condition (2) for $f \in \mathcal{S}$, one has the following necessary condition for $f \in \mathcal{U}$. This result has apparently appeared in [7] for a different context, but for the sake of completeness and a comparison in the sequel, we include the proof here as it is straightforward.

THEOREM 1.2 *Let $f \in \mathcal{U}$ have the form* (1). *Then* $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1$.

Proof The power series representation of f yields

$$|\mathcal{U}_f(z)| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| \leq 1, \quad z \in \mathbb{D}.$$

Letting $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right|^2 \mathrm{d}\theta \leq 1.$$

The desired inequality now follows by letting $r \to 1^-$. ■

THEOREM 1.3 *Let $f \in \mathcal{A}$ and has the form* (1). *If f satisfies the condition* (2), *then f is univalent in the disc $|z| < r_0 = \frac{1}{\sqrt{2}}$, and the radius is the best possible.*

Proof Consider the function g defined by $g(z) = r^{-1}f(rz)$, where $0 < r \leq 1$. Then

$$\frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n.$$

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The Cauchy–Schwarz inequality yields

$$\sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{n=2}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} = \frac{r^2}{1-r^2},$$

and since $r^2/(1-r^2) \leq 1$ for $0 < r \leq r_0 = \frac{1}{\sqrt{2}}$, it follows from Theorem 1.1 that $g \in \mathcal{U}$ (and hence univalent) in $\mathbb{D}$. This means that f is univalent in the disc $|z| < r_0$.

In order to prove that the radius of the disc is best possible, consider the function

$$f_0(z) = z - r_0 z^2, \quad r_0 = \frac{1}{\sqrt{2}}.$$

For this function,

$$\frac{z}{f_0(z)} = \frac{1}{1-r_0 z} = 1 + \sum_{n=1}^{\infty} r_0^n z^n$$

and therefore, with $b_n = r_0^n$,

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 = \sum_{n=2}^{\infty}(n-1)\left(\frac{1}{2}\right)^n = 1.$$

On the other hand, $\mathrm{Re} f_0'(z) = \mathrm{Re}(1-\sqrt{2}z) > 0$ for $|z| < r_0 = \frac{1}{\sqrt{2}}$, and $f_0'(r_0) = 0$ showing that f_0 is not univalent in any larger disc. ■

Theorem 1.4 *Let $f \in \mathcal{A}$ and has the form* (1). *If f satisfies the condition*

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1,$$

then the function g, defined by $g(z) = r^{-1}f(rz)$, belongs to $\mathcal{U}$ for $0 < r \leq r_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. In particular, f is univalent in the disc $|z| < r_0$ and the result is best possible.

Proof As g has the form

$$\frac{z}{g(z)} = \frac{rz}{f(rz)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n,$$

it follows that

$$\sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty} r^{2n}\right)^{\frac{1}{2}} \leq \frac{r^2}{\sqrt{1-r^2}},$$

which is less than or equal to 1 if $r^4 + r^2 - 1 \leq 0$, i.e. if $0 < r \leq r_0 = \sqrt{\frac{\sqrt{5}-1}{2}}$. This gives the desired conclusion.

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To prove sharpness, consider the function f_0 defined by

$$\frac{z}{f_0(z)} = 1 + \sum_{n=2}^{\infty} \frac{r_0^n}{n-1} z^n = 1 - r_0 z \log(1 - r_0 z).$$

It is easy to see that $\mathrm{Re}(z/f_0(z)) > 0$ for $z \in \mathbb{D}$ showing that $f_0(z) \neq 0$ for $0 < |z| < 1$. Now,

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 = \sum_{n=2}^{\infty} (n-1)^2 \frac{r_0^{2n}}{(n-1)^2} = \frac{r_0^4}{1 - r_0^2} = 1.$$

On the other hand, for $|z| < r_0$, we see that

$$\left| \left(\frac{z}{f_0(z)} \right)^2 f_0'(z) - 1 \right| = \left| \frac{-r_0^2 z^2}{1 - r_0 z} \right| < \frac{r_0^4}{1 - r_0^2} = 1,$$

while for $r_0 < z = r < 1$:

$$\left| \left(\frac{z}{f_0(z)} \right)^2 f_0'(z) - 1 \right|_{z=r} = \frac{r_0^2 r^2}{1 - r_0 r} > 1.$$

It follows that g_0 defined by $g_0(z) = r^{-1} f_0(rz)$ belongs to $\mathcal{U}$. That is, $|\mathcal{U}_f(z)| \leq 1$ holds in the disc $|z| < r_0$, but not in a larger one. Since

$$f_0'(z) = \frac{1 - r_0 z - r_0^2 z^2}{(1 - r_0 z)(1 - r_0 z \log(1 - r_0 z))^2}$$

and $f_0'(r_0) = 0$, then f_0 is not univalent in a larger disc than $|z| < r_0$. ∎

The above results can be extended to many general situations (see [8] and the references therein). For example to the class $\mathcal{U}(\lambda)$ of all functions $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition

$$|\mathcal{U}_f(z)| \leq \lambda \quad \text{for } z \in \mathbb{D},$$

and for some $\lambda \in (0, 1]$. As $\mathcal{U}(\lambda) \subset \mathcal{U} \subset \mathcal{S}$, functions in $\mathcal{U}(\lambda)$ are univalent in $\mathbb{D}$. The restriction on λ implies that functions in $\mathcal{U}(\lambda)$ are starlike in $\mathbb{D}$. Here a function $f \in \mathcal{A}$ is starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. The analytic conditions for starlikeness of $f \in \mathcal{S}$ can be written in the form

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D}.$$

It is worth pointing out that functions in the collection

$$\mathcal{L} = \left\{ z, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{1 \pm z + z^2} \right\}$$

are contained in $\mathcal{U} \cap \mathcal{S}^*$, and each function plays an important role in function theory, especially when considering the corresponding families $\mathcal{L}_g$ of close-to-convex functions f satisfying the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}$$

with $g \in \mathcal{L}$. To state our next result, let us recall the following result:

THEOREM A [8] *Any function* $f(z) := z + \sum_{n=2}^{\infty} a_n(f) z^n \in \mathcal{A}$ *satisfying*

$$\left|\mathcal{U}_f(z)\right| < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2}, \quad |z| < 1,$$

belongs to $\mathcal{S}^*$. *Moreover, there exists a non-starlike function* f *in* $\mathcal{U}$ *such that*

$$0 < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2} < \sup_{|z|<1} \left|\mathcal{U}_f(z)\right| \le 1 - |a_2(f)|.$$

We remark that although $\mathcal{U} \subset \mathcal{S}$, functions in $\mathcal{S}$ are not necessarily in $\mathcal{U}$. Thus, it is natural to consider some subsets of $\mathcal{S}$ which are included in $\mathcal{U}$. Our next result fulfils this aim.

THEOREM 1.5 *Let* $f \in \mathcal{A}$ *satisfy the condition*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda, \quad z \in \mathbb{D}, \tag{4}$$

for some $0 < \lambda \le 1$. *Then*

(1) $|\frac{z}{f(z)} - 1| < e^{\lambda} - 1$, $z \in \mathbb{D}$. (*Note that* $e^{\lambda} - 1 \le 1$ *whenever* $0 < \lambda \le \log 2$.)
(2) $f \in \mathcal{U}(\lambda')$, $\lambda' = (1+\lambda)e^{\lambda} - 1$.
(3) (*Note that* $\lambda' \le 1$ *whenever* $0 < \lambda \le \lambda_0 \approx 0.374823$, *where* λ_0 *is the root of the equation* $(1+\lambda)e^{\lambda} = 2$.)

In particular, if $f \in \mathcal{A}$ *and* $0 < \lambda \le \lambda_0 \approx 0.374823$, *then the following implications holds*:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda \Longrightarrow f \in \mathcal{U}.$$

Proof Set $p(z) = f(z)/z$. Then p is analytic in $\mathbb{D}$, $p(0) = 1$ and so, condition (4) is equivalent to

$$\frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} - 1 \prec \lambda z, \quad z \in \mathbb{D},$$

where $\prec$ denotes subordination and $p(z)$ is non-vanishing in $\mathbb{D}$. We can write

$$\frac{zp'(z)}{p(z)} = \lambda W(z), \quad z \in \mathbb{D},$$

where W is analytic in $\mathbb{D}$, $W(0) = 0$, and $|W(z)| < 1$ for $z \in \mathbb{D}$. Therefore,

$$\int_0^z \frac{p'(s)}{p(s)}\, \mathrm{d}s = \lambda \int_0^z \frac{W(s)}{s}\, \mathrm{d}s, \quad z \in \mathbb{D},$$

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from which we obtain

$$\log p(z) = \lambda \int_0^1 \frac{W(tz)}{t}\, \mathrm{d}t$$

so that $p(z) = \exp(\lambda\omega(z))$), where

$$\omega(z) = \int_0^1 \frac{W(tz)}{t}\, \mathrm{d}t$$

and ω is clearly analytic in $\mathbb{D}$, $\omega(0) = 0$, and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Therefore,

$$\frac{1}{p(z)} = \frac{z}{f(z)} = \exp(-\lambda\omega(z)), \quad z \in \mathbb{D}.$$

Also,

$$\left|\frac{z}{f(z)} - 1\right| \le \exp(\lambda|\omega(z)|) - 1 \le \exp(\lambda|z|) - 1 < \exp(\lambda) - 1$$

and the desired conclusion in part (1) follows.

For the proof of part (2), it suffices to observe that

$$\begin{aligned} \left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| &= \left| \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)} - 1\right) + \frac{zf'(z)}{f(z)} - 1 \right| \\ &\le \left|\frac{zf'(z)}{f(z)}\right| \left|\frac{z}{f(z)} - 1\right| + \left|\frac{zf'(z)}{f(z)} - 1\right| \\ &< (1+\lambda)(e^{\lambda} - 1) + \lambda = (1+\lambda)e^{\lambda} - 1, \end{aligned}$$

and the desired conclusion follows. ■

THEOREM 1.6 *Let* $\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma}$, $\phi(\sigma) = 2\zeta(2\sigma) - \zeta(2\sigma - 1)$ *and*

$$\psi(\sigma) = \frac{1}{2^{4\sigma+1}} + \zeta(2\sigma + 1) - 1 - \frac{1}{2^{2\sigma+1}}.$$

Assume that σ_0 *is such that* $\phi(\sigma_0) > 0$, *and* $C = C(\sigma_0) > 0$ *satisfies the condition*

$$\psi(\sigma_0) \le C \quad \text{and} \quad C \le \frac{(2^{\sigma_0 - 1} - 1)^2 - 3^{-2\sigma_0}}{2^{2(\sigma_0 - 1)}}. \tag{5}$$

Let $f \in \mathcal{S}$ *and define* F_σ *by*

$$\frac{z}{F_\sigma(z)} = \frac{z}{f(z)} \star \frac{\mathrm{Li}_\sigma(z)}{z},$$

where $\mathrm{Li}_\sigma(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\sigma}$ *and* $\star$ *denotes the usual convolution/Hadamard product of two convergent power series. Then* $F_\sigma \in \mathcal{U}$ *for* $\sigma \ge \sigma_0$.

Proof Let $f \in \mathcal{S}$ and set

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{6}$$

so that $z/F_\sigma(z)$ takes the form

$$\frac{z}{F_\sigma(z)} = 1 + \sum_{n=1}^{\infty} \frac{b_n}{(n+1)^\sigma} z^n.$$

The well-known area theorem [1, Theorem 11 on p. 193 of Vol. 2] gives

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \le 1, \tag{7}$$

and, the Cauchy–Schwarz inequality yields

$$\sum_{n=2}^{\infty} (n-1)\frac{|b_n|}{(n+1)^\sigma} \le \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} \frac{n-1}{(n+1)^{2\sigma}}\right)^{\frac{1}{2}}. \tag{8}$$

Now,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n-1}{(n+1)^{2\sigma}} &= \sum_{n=2}^{\infty} \frac{1}{(n+1)^{2\sigma-1}} - 2\sum_{n=2}^{\infty} \frac{1}{(n+1)^{2\sigma}} \\ &= \zeta(2\sigma-1) - 1 - \frac{1}{2^{2\sigma-1}} - 2\left(\zeta(2\sigma) - 1 - \frac{1}{2^{2\sigma}}\right) \\ &= 1 - \phi(\sigma). \end{aligned} \tag{9}$$

By hypothesis, $\phi(\sigma) \ge \phi(\sigma_0) > 0$ for $\sigma \ge \sigma_0$. Consequently, using (7) and (9), (8) implies that

$$\sum_{n=2}^{\infty} (n-1)\frac{|b_n|}{(n+1)^\sigma} \le 1 \quad \text{for } \sigma \ge \sigma_0.$$

It is worth pointing out that for the quantity $\mathcal{U}_{F_\sigma}(z)$ to be well-defined, we need to show that $\frac{z}{F_\sigma(z)} \ne 0$ in $\mathbb{D}$. Thus, by Theorem 1.1 and the last coefficient inequality, $F_\sigma \in \mathcal{U}$ if $\frac{z}{F_\sigma(z)} \ne 0$ for every $z \in \mathbb{D}$. In order to verify the non-vanishing condition, we use the representation of $\frac{z}{F_\sigma(z)}$ and obtain

$$\left|\frac{z}{F_\sigma(z)}\right| \ge 1 - \sum_{n=1}^{\infty} \frac{|b_n|\,|z|^n}{(n+1)^\sigma} > 1 - \frac{|b_1|}{2^\sigma} - \frac{|b_2|}{3^\sigma} - \sum_{n=3}^{\infty} \frac{|b_n|}{(n+1)^\sigma}. \tag{10}$$

Clearly it suffices to show that $\frac{z}{F_\sigma(z)} \ne 0$ only for $\sigma = \sigma_0$. In view of this observation, we first observe that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{|b_n|}{(n+1)^{\sigma_0}} &\le \left(\sum_{n=3}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty} \frac{1}{(n-1)(n+1)^{2\sigma_0}}\right)^{\frac{1}{2}} \\ &< \sqrt{1-|b_2|^2}\left(\frac{1}{2\cdot 4^{2\sigma_0}} + \sum_{n=4}^{\infty} \frac{1}{(n-1)^{2\sigma_0+1}}\right)^{\frac{1}{2}} \\ &= \sqrt{1-|b_2|^2}\sqrt{\frac{1}{2^{4\sigma_0+1}} + \zeta(2\sigma_0+1) - 1 - \frac{1}{2^{2\sigma_0+1}}} \\ &= \sqrt{1-|b_2|^2}\sqrt{\psi(\sigma_0)} \\ &\le \sqrt{C}\sqrt{1-|b_2|^2}, \end{aligned}$$

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where the last inequality is a consequence of the first condition in (5). Since $|b_1| = |-f''(0)/2| \le 2$ for each $f \in \mathcal{S}$, it follows from (10) that

$$\left|\frac{z}{F_{\sigma_0}(z)}\right| > 1 - \frac{1}{2^{\sigma_0 - 1}} - \frac{|b_2|}{3^{\sigma_0}} - \sqrt{C(1 - |b_2|^2)}.$$

Moreover, (7) implies that $|b_2| \le 1$. Let

$$g(x) = \frac{x}{3^{\sigma_0}} + \sqrt{C(1 - x^2)} \quad \text{for } 0 \le x \le 1.$$

It is a simple exercise to see that g has its maximum value of

$$g(x_0) = \frac{\sqrt{1 + 3^{2\sigma_0} C}}{3^{\sigma_0}}$$

at the point $x_0 = 1/\sqrt{1 + 3^{2\sigma_0} C}$. In view of this observation

$$\left|\frac{z}{F_{\sigma_0}(z)}\right| > 1 - \frac{1}{2^{\sigma_0 - 1}} - \frac{\sqrt{1 + 3^{2\sigma_0} C}}{3^{\sigma_0}},$$

which is non-negative whenever C and σ_0 are related by the second condition in (5). Finally, the condition $\frac{z}{F_{\sigma_0}(z)} \neq 0$ holds in $\mathbb{D}$ under the hypothesis. Thus, F_σ belongs to $\mathcal{U}$ for all $\sigma \ge \sigma_0$, and this completes the proof. ■

If $f \in \mathcal{S}$ has the form (6) with $b_1 = 0$, then the range of σ can be extended. However, we can quickly obtain the following corollary.

Corollary 1.7 *Let $f \in \mathcal{S}$ and define F_σ by*

$$\frac{z}{F_\sigma(z)} = \frac{z}{f(z)} \star \frac{\mathrm{Li}_\sigma(z)}{z},$$

where $\mathrm{Li}_\sigma(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\sigma}$. Then for $\sigma \ge 3/2$, $F_\sigma \in \mathcal{U}$, and hence F_σ is univalent in $\mathbb{D}$.

Proof Set $\sigma_0 = 3/2$ in Theorem 1.6. Then, $\zeta(3) \approx 1.20206$, and

$$\phi(3/2) = 2\zeta(3) - \zeta(2) = 2\zeta(3) - \frac{\pi^2}{6} \approx 0.75918,$$

$$\psi(3/2) = \zeta(4) - \frac{135}{128} = \frac{\pi^4}{90} - \frac{135}{128} \approx 0.0276357.$$

With $C = \psi(3/2)$, we find that

$$1 - \frac{1}{\sqrt{2}} - \frac{\sqrt{1 + 27C}}{3 \times \sqrt{3}} \approx 0.0385848,$$

and thus, all the required conditions of Theorem 1.6 are satisfied with $\sigma_0 = 3/2$. ■

We conclude this article with an open problem.

Problem Determine the smallest value of σ so that F_σ is either in $\mathcal{U}$ or in $\mathcal{S}$.

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